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We investigate several inverse methods to determine the dielectric profile of an almost periodic structure composed of several periodicities or tones. For widely spaced tones a resonance method provides the most useful solution while for closely spaced tones the Gelfand-Levitan-Marchenko theory appears to be the most promising. In this way these two methods complement each other so that a wide range of almost periodic media can be investigated. Criteria for characterizing resonant (widely spaced tones) and analytic (closely spaced tones) reflection coefficients are given and the appropriate inversion methods are demonstrated.

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INVERSE SCATTERING THEORY AND
ALMOST PERIODIC MEDIA

I. INTRODUCTION

A fundamental problem in physics is the investigation and description of an unknown object by probing or sensing with electromagnetic waves. Inverse scattering theory is concerned with the analytical and computational methods for solving this general problem. The physical model that will be considered in this report is the determination of the effective permittivity and density distribution of an inhomogeneous region by using the information contained in scattered electromagnetic waves.

Here we investigate the connection between two relatively diverse areas of mathematical physics, namely, one-dimensional inverse scattering theory and the theory of almost periodic functions. Our eventual objective is to determine an area of commonality and, more specifically, to discover inverse scattering methods which can be applied to waves scattered or reflected by almost periodic media. This communication serves the heuristic purpose of investigating the feasibility of our method, first suggested in [1]; the deeper mathematical questions will be addressed in a subsequent paper.

In the abstract setting of Sabatier [2], inverse scattering theory provides the mathematical methods for calculating a set C of physical parameters for a set S of scattering data. The idealized physical model that we will use is shown in Fig. 1. In this idealized model the set S consists of the incident plane wave e^{ikx} and the reflected wave $r(k)e^{-ikx}$, where k is the free-space wave number. The set C consists of μ_0, the permeability of free space, and $\epsilon(k, x)$, the permittivity. The inverse scattering problem here is to determine the unknown permittivity $\epsilon(k, x)$ from the reflection coefficient $r(k)$. In general, inverse problems start with limited *a priori* knowledge of the set C for the inhomogeneous region; this *a priori* knowledge and the scattering data that are available will suggest the particular inversion method that should be used and the physical parameters that can be reconstructed. In contrast direct problems start with complete knowledge of the set of physical parameters and seek to determine the scattering data. Clearly most practical problems will use a hybrid approach. For example, an inverse problem has traditionally been solved by the iterative solution of the relevant direct problem until the calculated scattering data agree with the experimental data.

The mathematical foundation for one-dimensional inverse scattering theory was laid by the Soviet mathematicians Gel'fand, Levitan and Marchenko, circa 1950 [3,4], for use in quantum mechanical problems. Subsequent work by Kay and Moses [5,7], and Newton [6] in the United States extended the availability of these results and expanded the theory to include the electromagnetic problem. Of particular interest is the treatment by Kay [5] in which several examples display the usefulness of the Gel'fand-Levitan-Marchenko theory. Much of the initial work [5,7,8,9,10] emphasized exact, closed-form solutions to the problem when the scattered data were represented by rational functions. Numerical [11,12,13] and approximate [14,15] solutions provided additional understanding of this problem. A recent summary of the general status and applications of several inverse scattering methods can be found in a recent special issue of the *IEEE Transactions on Antennas and Propagation* [16] and in a future issue of *Radio Science* [17].

The theory of almost periodic functions has its origins in the work of Bohr [18] and Besicovitch [19], performed during the 1920's and 1930's. Important contributions were subsequently made by Wiener [20] and others [21]. More recently, these functions have been applied to direct problems in mechanics [22,23] and electromagnetic wave propagation [24,25,26,27]. These functions are useful in describing certain types of imperfections in periodic structures and so may play a role in solid state physics and in the design of optical and microwave devices.

Here we investigate several methods to determine the profile of a structure where we have the *a priori* knowledge that this structure is described by an almost periodic function. Trubowitz [28] has presented the inverse scattering theory for periodic potentials. After reviewing the necessary facts needed from inverse scattering theory, the theory of almost periodic functions and reflection from almost periodic media, we use several examples to illustrate the application of inverse scattering theory to reflection from almost periodic media. Finally, we give a brief discussion and several general conclusions resulting from our investigation.

II. MATHEMATICAL BACKGROUND

A. Inverse Scattering

In a source-free inhomogeneous region the electric and magnetic field vectors $\underline{E}$ and $\underline{H}$ satisfy the time-harmonic differential equations

$$\nabla^2 \underline{E} + \nabla\left[\underline{E} \cdot \frac{\nabla\epsilon}{\epsilon}\right] k^2 \epsilon \underline{E} = 0, \tag{1}$$

$$\nabla^2 \underline{H} + \frac{\nabla\epsilon}{\epsilon} \times \nabla \times \underline{H} + k^2 \epsilon \underline{H} = 0, \tag{2}$$

In the physical model of Fig. 1, we have assumed that the plane-polarized electromagnetic field is normally incident on a semi-infinite inhomogeneous region whose permittivity $\epsilon(k, x)$ is assumed to obey the dispersion relation

$$\epsilon(k, x) = \begin{cases} \epsilon_0 \left(1 - \frac{1}{k^2} q(x)\right) & (x \geq 0) \\ \epsilon_0 & (x < 0) \end{cases} \tag{3}$$

where ϵ_0 is the permittivity of free space, so that the behavior of the time-harmonic transverse electric wave amplitude $u(k, x)$ is described by the differential equation

$$\frac{d^2}{dx^2} u(k, x) + \left[1 - \frac{1}{k^2} q(x)\right] u(k, x) = 0. \tag{4}$$

The transverse magnetic field case can be transformed to a differential equation with the same functional form. Here $q(x)$ is a profile function related to the density of the unknown region, where $q(x)$ is assumed to be piecewise continuous. Then $r(k)$ will be an analytic function of k so that the conservation of energy condition is expressed as

$$|r(k)|^2 \leq 1. \tag{5}$$

A physical description of the inverse scattering procedure [7] is obtained by using the Fourier transform of the differential Eq. (4),

$$\frac{\partial^2}{\partial x^2} U(x, t) - \frac{\partial^2}{\partial t^2} U(x, t) = q(x) U(x, t). \tag{6}$$

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The incident plane wave is represented as an incident impulse

$$U_{inc}(x, t) = \delta(x - t) \tag{7}$$

which produces a reflected transient

$$R(x + t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} r(k) e^{-ik(x - t)} \, dk - i \sum_{\nu} r_{\nu} e^{-ik_{\nu}(x - t)} \tag{8}$$

where the r_{ν} in the second term are the residues at the poles, if any, of $r(k)$ on the positive imaginary k-axis. Because of causality, we must have

$$R(x + t) = 0 \qquad (x < t) \tag{9}$$

i.e., a reflected transient is not produced until the incident pulse has interacted with the medium. It is possible (see, e.g., Kay [5,7]) to relate the wave amplitude, $U(x, t)$, in the inhomogeneous region, $x > 0$, with the wave amplitude in the free space region, $U_0(x, t)$, $x \leqslant 0$, by the transformation

$$U(x, t) = U_0(x, t) + \int_{-\infty}^{\infty} K(x, z) \, U_0(z, t) \, dz \tag{10}$$

where

$$U_0(x, t) = \delta(x - t) + R(x + t) \quad (x \leqslant 0).$$

From physical considerations we know that $U(x, t)$ is a right-moving transient that is,

$$U(x, t) = 0 \qquad (x > t) \tag{11}$$

so that $U(x, t)$ depends only on the medium traversed by $U_0(x, t)$ up to time t. Therefore, $K(x, z) = 0$ for $z > x$. Substituting the expression for the field in the free-space, $U_0(x, t)$, gives the integral equation

$$R(x, t) + K(x, t) + \int_{-y}^{x} K(x, z) \, R(z + t) \, dz = 0 \quad (x > t) \tag{12}$$

which is Kay's version of the Gel'fand-Levitan-Marchenko integral equation and is to be solved for the function $K(x, t)$. Conditions of $K(x, t)$ are found by substituting the expression (10) for $U(x, t)$ in the differential (Eq. (6)); the function $K(x, t)$ satisfies the same differential equation as $U(x, t)$,

$$\frac{\partial^2}{\partial x^2} K(x, t) - \frac{\partial^2}{\partial t^2} K(x, t) = q(x) K(x, t), \tag{13}$$

subject to the conditions

$$K(x, -x) = 0 \tag{14}$$

and

$$\frac{d}{dx} K(x, x) = 2 q(x). \tag{15}$$

From condition (15) we see that the inverse scattering problem is solved for $q(x)$ if we can solve the integral equation (12) that relates the unknown function $K(x, t)$ with the scattering data $R(x + t)$.

Thus the solution of the one-dimensional inverse scattering problem that we are investigating has been reduced to the solution of Eq. (12) together with the condition (15). Several exact and approximate methods of solving Eq. (12) have been reviewed by Jordan [29]; the most important for obtaining closed-form solutions useful for engineering applications is that of Kay [7], where $r(k)$ is a rational function of the wavenumber k,

$$r(k) = r_0 \frac{(k - \mu_1) (k - \mu_2) \dots}{(k - k_1) (k - k_2) \dots}, \tag{16}$$

where k_1, k_2, ... are the poles and μ_1, μ_2, ... are the zeros of $r(k)$ in the complex k-plane. Examples of these solutions have been given by Jordan et al. [9,10,11] for several pole-zero configurations; examples of numerical and approximate solutions have been given by Jaggard et al. [13,15] and Krueger [12].

B. Almost Periodic Functions

In the restricted sense in which we use it here, an almost periodic function is one which possesses a discrete or line spectrum. This implies that such a function $f(x)$ can be written in a generalized Fourier series

$$f(x) = \sum_n a_n e^{i\kappa_n x} \tag{17}$$

where κ_n is the wavenumber or spatial frequency of the nth harmonic and a_n is its amplitude. (Often we will refer to κ_n as the tone frequency and a_n as the tone strength.) This equation can be inverted to produce the harmonic amplitudes.

$$a_n = \lim_{X \to \infty} \frac{1}{2X} \int_{-X}^{X} f(x) e^{-i\kappa_n x} \, dx. \tag{18}$$

Clearly, Eq. (17) represents a periodic function if only one tone is present or if all of the tone frequencies are commensurable. In this case Eq. (17) and (18) become the usual expressions for the Fourier series and Fourier inversion formula, respectively.

Although almost periodic functions do not possess a period, as noted above, they do possess an almost period or translation number τ such that

$$|f(x + \tau) - f(x)| < \sigma, \tag{19}$$

where $\tau = \tau(\sigma)$. This relation provides an alternative definition to the more restrictive one expressed by Eq. (17). Another property of interest is the Parseval relation expressed as

$$\lim_{X \to \infty} \frac{1}{2X} \int_{-X}^{X} |f(x)|^2 \, dx = \sum_n |a_n|^2. \tag{20}$$

These and other characteristics can be shown from the defining equations (Eqs. (17) and (19)) and can be found in the literature [18,19,20,23].

C. Reflection from Almost Periodic Media

If the potential function $q(x)$ is almost periodic an approximate theory can be developed for the reflection coefficient using the method developed in [26,27,30]. Here we examine the three tone case described by

$$q(x) = \begin{cases} -\dfrac{\eta\kappa^2}{4} \left[\dfrac{m}{2} \cos\left[(\kappa + \Delta)x\right] + \cos\left[\kappa x\right] + \dfrac{m}{2} \cos\left[(\kappa - \Delta)x\right] \right] & (0 \leqslant x \leqslant l) \\ 0 & (x < 0,\ x > l) \end{cases} \tag{21}$$

This function represents a modulated periodic structure with carrier wavenumber κ and sideband wavenumbers $\kappa \pm \Delta$. Clearly Δ is the tone spacing. Here η is the relative amplitude of the carrier and m is the modulation index. By assuming an appropriate form for the electric field amplitude,

$$u(x, k) = F_0(x)\, e^{i(\kappa/2)x} + B_-(x)\, e^{-i(\frac{\kappa}{2} - \Delta)x}$$

$$+ B_0(x)\, e^{-i(\kappa/2)x} + B_-(x) e^{-i(\frac{\kappa}{2} - \Delta)x} \tag{22}$$

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in the vicinity of the primary Bragg resonance defined by

$$k = \kappa/2 \tag{23}$$

the following vector coupled mode equations (24) result under suitable approximations. These include small or moderate amplitudes ($\eta \leq 0.5$, $m \leq 1$) and relatively closely spaced tones ($\Delta \leq \kappa/4$) as discussed in [25,26].

$$\underline{F}' - i\,\delta_F\,\underline{F} = i\,\chi\,\underline{B}$$
$$-\underline{B}' - i\,\delta_B\,\underline{B} = i\,\chi^\dagger\,\underline{F} \tag{24}$$

where

$$\underline{F} = [F_0] \qquad \underline{B} = \begin{bmatrix} B_- \\ B_0 \\ B_+ \end{bmatrix}$$

$$\delta_F = [\delta_0] \qquad \delta_B = \begin{bmatrix} \delta_- & 0 & 0 \\ 0 & \delta_0 & 0 \\ 0 & 0 & \delta_+ \end{bmatrix}$$

$$\chi = [\chi_- \quad \chi_0 \quad \chi_+]$$

and

$$\delta_0 = \Delta k = \omega/c - \kappa/2 \qquad \delta_\pm = \delta_0 \pm \Delta$$
$$\chi_- = \chi_+ = \eta m \,\kappa/16 \qquad \chi_0 = \eta\kappa/8.$$

The prime (') indicates differentiation with respect to x and the dagger ($\dagger$) denotes the Hermitean transpose. A constant loss can be added by letting δ_0 and $\delta_\pm$ become complex.

By applying the appropriate boundary conditions at the slab ends,

$$\underline{F}(0) = [1] \qquad\qquad \underline{F}(l) = [\tilde{T}] \tag{25}$$

$$\underline{B}(0) = \begin{bmatrix} R_- \\ R_0 \\ R_+ \end{bmatrix} = \mathbf{R}^{\dagger*} \qquad \underline{B}(l) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

the problem is completely specified. Here the asterisk (*) denotes complex conjugate and the last expression in Eq. (25) defines the reflection matrix $\mathbf{R}$ while the second expression defines the scalar transmission coefficient $\tilde{T}$. It is preferable to reformulate the above two-point boundary-value problem, defined by Eqs. (23)-(24), into an initial-value problem for the reflection matrix to avoid problems of numerical instability or formidable algebra. This leads to the matrix Riccati equation

$$\mathbf{R}' = i\chi + i\delta_F\,\mathbf{R} + i\mathbf{R}\,\delta_B + i\,(\mathbf{R}\,\chi^\dagger)\,\mathbf{R} \tag{26}$$

with initial condition

$$\mathbf{R}(0) = [0 \quad 0 \quad 0]. \tag{27}$$

This formulation is particularly amenable to numerical solution and can be used for any number of tones. A result for the three tone case corresponding to (21) is shown in Fig. 2. Here the scalar reflection $\tilde{R}$ is given by

$$\tilde{R} = \mathbf{R}\mathbf{R}^\dagger \tag{28}$$

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and is plotted for a single value of normalized coupling

$$\chi l = \sqrt{(\chi_0 l)^2 + [(\chi_+ + \chi_-)l]^2} = 2 \tag{29}$$

as a function of normalized frequency $\tilde{f} = 2\Delta k/\chi$ for various tone spacings and losses. Note particularly the large peak at the primary Bragg resonance defined by Eq. (23), or $\tilde{f} = 0$, and the secondary Bragg resonances at

$$k = (\kappa \pm \Delta)/2. \tag{30}$$

These results suggest the following observations:

(1) For widely spaced tones (bottom plots of Fig. 2), the carrier and modulation frequencies tend to act independently. Indeed, this can be shown mathematically since the vector coupled mode equations separate into three scalar coupled mode equations.

(2) For closely spaced tones (top plots of Fig. 2), the action of the various tones is no longer independent but these plots are suggestive of the rational function reflection coefficients displayed in [7,8,9].

For these reasons it is natural to examine these two classes separately in the inverse problem. For widely spaced tones, use of resonance theory seems to be the most promising approach while for closely spaced tones the use of Gel'fand-Levitan-Marchenko theory appears to be most promising. In this way the two methods complement each other so that a wide range of almost periodic media may be characterized.

III. INVERSION METHODS FOR ALMOST PERIODIC MEDIA

A. Widely Spaced Tones

For extremely widely spaced tones ($\Delta \gg \chi$) in which the tones act independently, the results of scalar coupled mode theory can be used in the vicinity of each Bragg resonance. For example, if an N tone medium is defined by

$$q(x) = \begin{cases} -\frac{1}{4} \sum_{i=1}^{n} n_i \kappa_i^2 \cos \kappa_i x & (0 \leqslant x \leqslant l) \\[2mm] 0 & (x < 0, \ x > l) \end{cases} \tag{31}$$

where $|\kappa_j - \kappa_i| \ll \kappa_i$ or κ_j $(i, j = 1, 2, \ldots N)$, then the scalar reflection R_i in the vicinity of $k = \kappa_i/2$ is given by [31]

$$R_i = \frac{i\kappa_i l}{\sqrt{\chi_i^2 - \delta_i^2} \ l \coth \sqrt{\chi_i^2 - \delta_i^2} \ l - i\,\delta_i l} \tag{32}$$

where

$$\chi_i = \eta_i \kappa_i/8 \tag{33}$$

$$\delta_i = k - \kappa_i/2. \tag{34}$$

The peak value is

$$|R_i|_{max} = \tanh (\chi_i l) \tag{35}$$

which can be inverted to get all of the $\chi_i l$'s. The placement of the peak gives the values of the κ_i's.

If the combination $\chi_i l$ is not desired, but instead the values for both χ_i and l are to be determined, one needs to examine the Bragg resonances at the sum and difference wavenumbers defined by

$$k = \pm m\,\kappa_1/2 \pm n\,\kappa_2/2 \pm p\,\kappa_3/2 \pm \ldots \tag{36}$$

where $m, n, p, \ldots$ are positive integers or zero. As the simplest example, we briefly consider the two-tone care where $N = 2$ in Eq. (31). Then in addition to the values of κ_1, κ_2 and χ_1 and χ_2 one can look at the sum and difference resonances which occur at $k = (\kappa_2 + \kappa_2)/2$ and $k = |\kappa_2 - \kappa_1|/2$ respectively. The peak values of the reflection coefficient here are determined by second order coupled mode theory as described in [32]. The result is

$$|R_{\text{sum}}|_{\text{max}} = \tanh\,(\chi_{\text{sum}}l) \tag{37}$$

and

$$|R_{\text{diff}}|_{\text{max}} = \tanh\,(\chi_{\text{diff}}l) \tag{38}$$

where

$$\chi_{\text{sum}} = \eta_1\,\eta_2\,(\kappa_1 + \kappa_2)/8 \tag{39}$$

$$\chi_{\text{diff}} = \eta_1\,\eta_2\,|\kappa_1 - \kappa_2|^3/2\,(\kappa_1 + \kappa_2). \tag{40}$$

By using these values of $\chi_1 l$, $\chi_2 l$, $\chi_{\text{sum}}l$ and $\chi_{\text{diff}}l$ from Eqs. (32), (37) and (38), the values of η_1, η_2, and l can be found using Eqs. (33), (39) and (40). In this way the original potential function $q(x)$ has been reconstructed up to a relative phase between periodicities.

Referring to Fig. 3 the steps for reconstructing the two tone potential are as follows:

(i)   Find the primary maxima evidenced by large reflections and evaluate κ_1, κ_2, $\chi_1 l$ and $\chi_2 l$ from the reflection coefficient plot and Eq. (35).

(ii) Find the secondary maxima at $k = |\kappa_2 \pm \kappa_1|/2$ and find $\chi_{\text{sum}}l$ and $\chi_{\text{diff}}l$ from Eqs. (37)-(38). Calculate η_1, η_2 and l from Eqs. (33), (39) and (40).

The method can be extended to the three-tone modulated periodicity of (21) and to potentials with an arbitrary number of tones.

The criterion for the applicability of this method is that the tones are separated sufficiently so that they act independently. This occurs when the bandgaps due to each periodicity are spaced in frequency so that they do not coalesce. This condition is described by the relative balance of the periodicity amplitudes with the wavenumbers as expressed by

$$|\eta_i|^2 \ll 16\,|\kappa_i - \kappa_j| \qquad (i, j = 1, 2, \ldots N,\ i \neq j). \tag{41}$$

B. Closely Spaced Tones

If the tones are closely spaced so that condition (41) does not hold, then the simplified expression (Eq. (32)) for the reflection coefficient cannot be used. The traditional approach of iterating between the direct problem, via the matrix Riccati equation, and the experimental data leads to formidable algebra and a problem of physical interpretation. Alternatively, the inverse problem for closely spaced tones can be approached by observing that the graph of $|r(k)|^2$ for a rational reflection coefficient with zeros as well as poles resembles the behavior of the scalar reflection $\tilde{R}^2$ for reflection from an almost-periodic medium with three closely spaced tones (Fig. 2, second plot, $\delta = 2\chi$, $Ll = 0.4$). The bandpass characteristics of the almost-periodic reflection coefficient in the vicinity of the central Bragg resonance suggests that poles on the unit circle (Butterworth poles) could be used as a first approximation. The number of "sidelobes" of $|r(k)|^2$ will be determined by the number of zeros. The rapid damping of $|r(k)|^2$ as $k \to \infty$ can be accounted for by absorption in the inhomogeneous region.

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Combining these observations suggests that a rational reflection coefficient with the form

$$r(k) = \frac{k_1 k_2 k_3}{\mu_1} \cdot \frac{k - \mu_1}{(k - k_1)(k - k_2)(k - k_3)} \qquad (42)$$

where the locations of the poles, k_1, k_2, k_3, and the zero, μ_1, in the complex k-plane will be chosen to meet the requirement for energy conservation, Eq. (5), as well as approximating the almost-periodic reflection coefficient. This particular choice for $r(k)$ was obtained by combining the results of the solutions to two different inverse scattering problems; namely, the case of the reflection coefficient, $r_\delta(k)$, due to a complex potential function [33] and the case of a 2-pole Butterworth function reflection coefficient, $r_B(k)$ [8]. Thus we write $r(k)$ as

$$r(k) = r_B(k) \cdot r_\delta(k) \qquad (43)$$

where

$$r_B(k) = \frac{k_1 k_2}{(k - k_1)(k - k_2)}$$

and

$$r_\delta(k) = \frac{k_3}{\mu_1} \frac{k - \mu_1}{k - k_3}.$$

Here we set

$$k_1 = \frac{\sqrt{2}}{2}(1 - i) \qquad k_2 = -k_1^*$$

$$k_3 = ia \qquad \mu_1 = i(a + \delta) \qquad (0 < \delta \ll a)$$

so that

$$|r(k)|^2 = \frac{1}{k^4 + 1} \cdot \frac{a^2}{(a + \delta)^2} \cdot \frac{k^2 + (a + \delta)^2}{k^2 + a^2} \qquad (44)$$

which has the general characteristics of the plot of $\bar{R}^2$ being considered. These analytical results will aid in the solution of the inverse problem for almost periodic media; this physical model can be interpreted as an inhomogeneous medium whose effective permittivity has dispersive and dissipative characteristics which we now determine.

Applying the analysis of [33] for potentials with weak absorption, we write

$$\bar{q}(x) = q_B(x) + [q_\delta(x) + ikp_\delta(x)] \cos \kappa x \qquad (45)$$

where $\bar{q}(x)$ is the effective complex profile function for almost periodic medium. Using the method of [8] on the $r_B(k)$ of (43), we find

$$q_B(x) = \frac{4}{(1 + \sqrt{2}x)^2} \qquad (46)$$

using the method of [33] on $r_\delta(k)$, we find

$$q_\delta(x) = 2 [R_\delta(2x)]' [F(x)] + 3 [R_\delta(2x)]^2 [F(x)]^4 \qquad (47)$$

$$p_\delta(x) = R_\delta(2x) [F(2x)]^2 \qquad (48)$$

where

$$R_\delta(x) = \begin{cases} a\,\delta(1 - \delta)\,e^{-a(1-\delta)x} & (x \geq 0) \\ \\ 0 & (x < 0) \end{cases} \qquad (49)$$

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$$F(x) = [1 - \delta\, e^{-2a(1-\delta)x}]^{-1/2}. \tag{50}$$

The inverse scattering method has been demonstrated with a reflection coefficient with three poles and one zero. More complicated pole-zero configuration can be used to obtain closer agreement with the experimental $|r(k)|^2$ and to model profiles with more tones. Losses in the dielectric region are readily included in the inversion method by poles and zeros on the positive imaginary axis.

IV. DISCUSSION

Inverse scattering theory has been used to determine the permittivity profile of a one-dimensional inhomogeneous medium by analyzing the properties of the reflection coefficient amplitude as a function of incident wavenumber. The problem of scattering from an almost periodic medium can be analyzed by considering two cases which are dependent on the relative spacing of the tones.

If the reflection coefficient is composed of several widely spaced primary maxima then the resonance method for widely spaced tones should be used. The number and spacing of the tones can be determined from the number and spacing of the reflection maxima. A check on these tone characteristics can be made by examining secondary maxima. The length and amplitude of the various perturbations can be found from the heights of the various relative maxima of the reflection data.

If the reflection coefficient is composed of closely spaced maxima, then the complementary inversion method must be used. Here the reflection coefficient is approximated by a rational function of wavenumber with respect to the principal maxima. The number of poles and zeros of this reflection coefficient function and their location in complex wavenumber space are varied until this function approximates the reflection data. Then using the inversion algorithm of Kay, based on the work of Gel'fand, Levitan and Marchenko, the potential or permittivity is reconstructed.

Future work will be concerned with the more accurate approximation of reflection data and with finding the sensitivity of these methods to reflection data errors.

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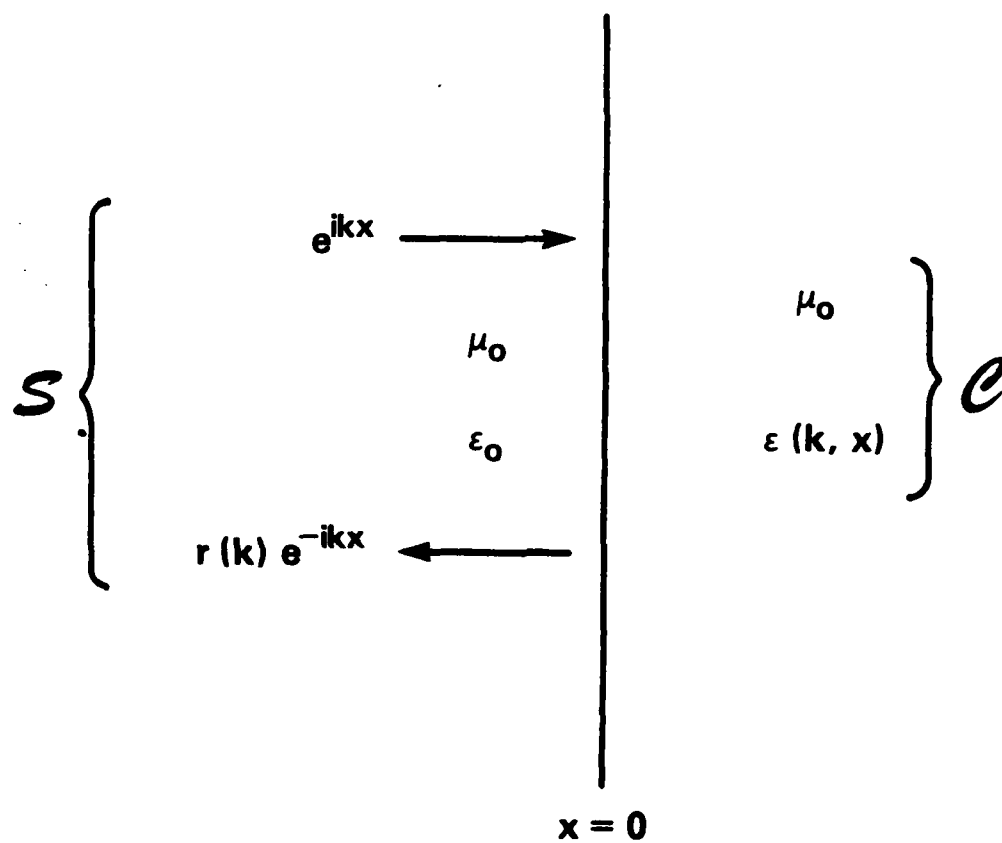
Fig. 1 — IDEALIZED PHYSICAL MODEL One dimensional scattering of a time-harmonic wave
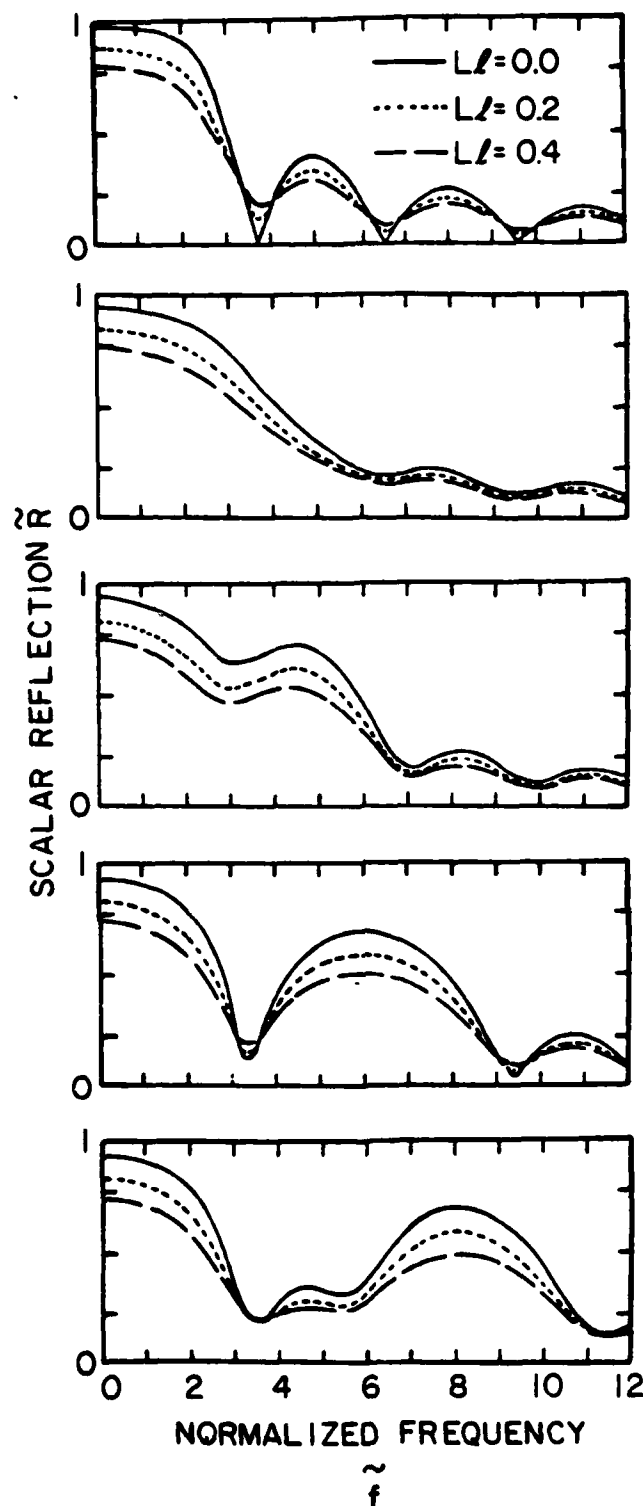of unit amplitude from a half space characterized by $\epsilon(k, x)$

Fig. 2 — REFLECTION COEFFICIENTS FOR THREE TONE ALMOST PERIODIC MEDIUM. Plots of scalar reflection $\tilde{R}$ as a function of normalized frequency $\tilde{f}$ for various tone spacings Δ for a three-tone slab of length l as described by (21) The tone spacings are $\delta = 0$, 2χ, 4χ, 6χ, and 8χ top to bottom (see text) and the relative loss is Ll. Therefore, the top two plots represent closely spaced tones while the bottom two plots display tones of moderate or wide spacing. The modulation is constant $(m = 1)$ and $\chi l = 2$. Adapted from [27].
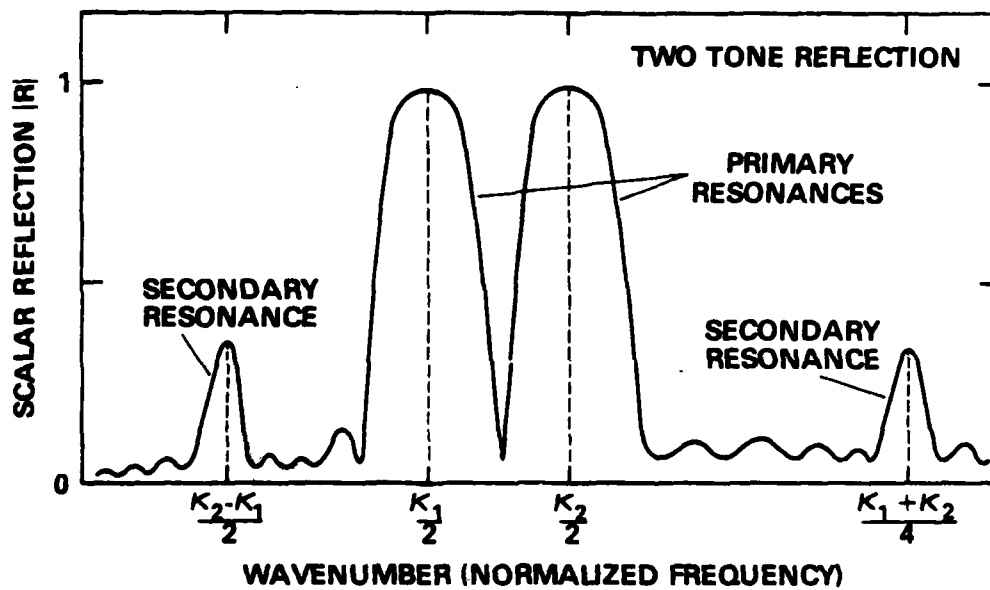
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Fig. 3 — TWO TONE REFLECTION COEFFICIENT. For moderate or widely spaced tones with wavenumber κ_1 and κ_2. primary Bragg resonances occurs at $k = \kappa_1/2$ and $\kappa_2/2$. Secondary resonance occurs at subharmonic and superharmonic wavenumbers as shown.

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